Blacktown Boys' High School

2022

HSC Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided for this paper
- All diagrams are not drawn to scale
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 12)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Assessor: X. Chirgwin

Student Name: ____

Teacher Name: ____

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2022 Higher School Certificate Examination.

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1–10.

Q1 Which of the following is the complex number $-3-3\sqrt{3}i$?

A. $6e^{-\frac{i\pi}{3}}$

B. $6e^{\frac{i\pi}{3}}$

C. $6e^{-\frac{2i\pi}{3}}$

D. $6e^{\frac{2i\pi}{3}}$

Q2 What is the Cartesian equation of the line $\underset{\sim}{r}=\begin{pmatrix}1\\5\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\end{pmatrix}$?

A. $2x+3y-17=0$

B. $3x+2y+13=0$

C. $2x-3y-13=0$

D. $3x-2y+17=0$

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Q3 Using a suitable substitution $\int_1^{e^5}\frac{(\log_e x)^5}{x}dx$ may be expressed in terms of u as

A. $\int_1^5\frac{(\log_e u)^5}{u}du$

B. $\int_0^5\frac{u^5}{e^u}du$

C. $\int_1^{\log_e 5}u^5du$

D. $\int_0^5 u^5du$

Q4 Let $x\in\mathbb{Z}$. The contrapositive of the statement "If x is even, then $5x-11$ is odd." is

A. If x is odd, then $5x-11$ is even.

B. If $5x-11$ is even, then x is odd.

C. If x is odd, then $5x-11$ is odd.

D. If $5x-11$ is odd, then x is even.

Q5 Which of the following is the expression for $\int\sin^3 x\,dx$?

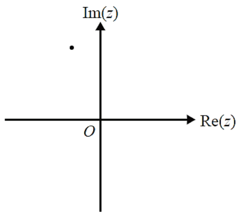
A. $\frac{1}{3}\cos^3 x-\cos x+C$

B. $\frac{1}{3}\cos^3 x+\cos x+C$

C. $\frac{1}{3}\sin^3 x-\sin x+C$

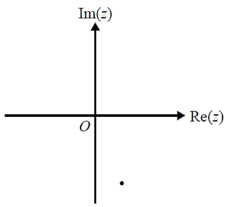
D. $\frac{1}{3}\sin^3 x+\sin x+C$

Q6 A particular complex number z is represented by the point on the following argand diagram.

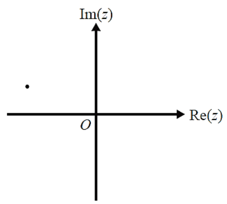


The complex number $-i\,\overline{z}$ is best represented by?

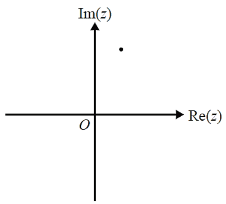
A.



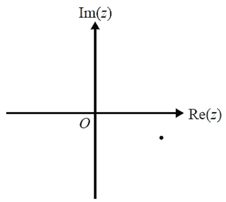
B.



C.



D.



Q7 If $2i - 3$ is a root of the polynomial equation $5x^3 + mx^2 + 59x - 13 = 0$, where m is real, then the value of m is

A. 31

B. -31

C. 29

D. -29

Q8 It is given that a, b are real, and c, d are purely imaginary. Which pair of inequalities must always be true?

A. $a^2c^2 + b^2d^2 \geq 2abcd,\quad a^2b^2 + c^2d^2 \leq 2abcd$

B. $a^2c^2 + b^2d^2 \geq 2abcd,\quad a^2b^2 + c^2d^2 \geq 2abcd$

C. $a^2c^2 + b^2d^2 \leq 2abcd,\quad a^2b^2 + c^2d^2 \leq 2abcd$

D. $a^2c^2 + b^2d^2 \leq 2abcd,\quad a^2b^2 + c^2d^2 \geq 2abcd$

Q9 A body is moving in a straight line and, after t seconds, it is x metres from the origin and travelling at $v\ ms^{-1}$. Given that $v = x$, and that $t = 5$ where $x = -1$, the equation for x in terms of t is

A. $x = \sqrt{2t - 9}$

B. $x = -\sqrt{2t - 9}$

C. $x = e^{5-t}$

D. $x = -e^{t-5}$

Q10 Without evaluating the integrals, which one of the following integrals is greater than zero?

A. $\displaystyle\int_{-1}^{1} \left(e^{-x^2} - 2\right) dx$

B. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^5 \sin x \, dx$

C. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^3}{5 + \cos x} dx$

D. $\displaystyle\int_{-2}^{2} \tan^{-1}(x^3) \, dx$

End of Section I

Section II

90 Marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question on a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Given that $z = \sqrt{3} - i$

i) Express z in modulus-argument form. **2**

ii) Use De Moivre's theorem to evaluate z^7, and leave your answer in the form $x + iy$. **2**

iii) Use De Moivre's theorem to evaluate $\dfrac{z^7}{(\bar{z})^7}$, and leave your answer in the form $x + iy$. **2**

b) Let $\underset{\sim}{p} = \frac{2}{3}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$, $\underset{\sim}{q} = -\frac{1}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$ and $\underset{\sim}{r} = \frac{2}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} - \frac{1}{3}\underset{\sim}{k}$

be vectors in 3-dimensional space defined by perpendicular unit vectors

$\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$.

i) Show that $\underset{\sim}{p}$ is a unit vector. **1**

ii) Show that $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$ are perpendicular to each other. **2**

iii) Given that $\underset{\sim}{s} = \underset{\sim}{p} + \underset{\sim}{q} + \underset{\sim}{r}$, show that $\underset{\sim}{s} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$. **1**

iv) Given that $\underset{\sim}{t} = \frac{2}{\sqrt{3}}\underset{\sim}{p} - \sqrt{3}\underset{\sim}{q} + \frac{1}{\sqrt{3}}\underset{\sim}{r}$, find $\underset{\sim}{s} \cdot \underset{\sim}{t}$. **1**

c) Show that $\displaystyle\int_0^1 \frac{2 - x}{(1 + 5x^2)(1 + 10x)}\,dx = \frac{1}{10}\log_e \frac{121}{6}$ **4**

End of Questions 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) Find

i) $\displaystyle\int \frac{1}{x}(1 + \log_e x)^6 dx$ **2**

ii) $\displaystyle\int \frac{1}{\sqrt{2 + x} + \sqrt{x}} dx$ **2**

iii) $\displaystyle\int \sin^3 x \cos^2 x \, dx$ **3**

iv) $\displaystyle\int \frac{2x + 7}{x^2 + 10x + 29} dx$ **3**

b) Let $P(x) = x^4 + ax^3 - 40x^2 + 41x + b$ where a and b are real numbers.

It is known that $x = 7$ and $x = \dfrac{1 - i\sqrt{3}}{2}$ are zeros of $P(x)$.

i) Explain why $x^2 - x + 1$ must be a factor of $P(x)$. **2**

ii) Find a and b. **3**

End of Questions 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) Shade on the Argand diagram the region given by **3**

$$|z-2| \le 2 \;\cap\; -\frac{\pi}{4} \le \arg z < \frac{\pi}{6}$$

b) Consider the following two vectors, where p is a scalar constant, **3**

$$\underset{\sim}{a} = 6\underset{\sim}{i} + p\underset{\sim}{j} - 13\underset{\sim}{k} \text{ and } \underset{\sim}{b} = (1-4p)\underset{\sim}{i} + (p+3)\underset{\sim}{j} + 6\underset{\sim}{k},$$

Find the values of p if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

c) A particle of mass m is moving in a straight line under the action of a force **3**

$$F = \frac{m}{x^3}(4-7x)$$

Find the particle's velocity $v\ ms^{-1}$ in terms of its displacement x, if the particle has a velocity of $-1\ ms^{-1}$ when the particle is 1 metre to the right of the origin.

d) Use integration by parts to find $\int e^x \cos 10x\, dx$ **3**

e) i) Suppose $f(x) = \sqrt{1+x}$, show that $f'(x) < \frac{1}{6}$ for $x > 8$. **1**

ii) Hence, show that $\sqrt{1+x} \le 3 + \frac{x-8}{6}$ when $x \ge 8$. **2**

End of Questions 13

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Question 14 (15 marks) Use a SEPARATE writing booklet.

a) With respect to a fixed origin O, the straight lines L_1 and L_2 have respective vector equations

$$\underset{\sim}{r_1} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where λ and μ are scalar parameters.

The points A and C lie on L_1 and L_2, where $\lambda = 0$ and $\mu = 0$, respectively.

i) Find $|\overrightarrow{AC}|$, in exact surd form. **2**

ii) Show that L_1 and L_2 intersect at some point B and find its coordinates. **2**

iii) Find the size of the angle θ, between L_1 and L_2. **1**

iv) The point D is such so that $ABCD$ is a kite. Show that the area of the kite is $16\sqrt{3}$ square units. **2**

b) The velocity, $v\ ms^{-1}$, of a particle moving in Simple Harmonic Motion along the x-axis is given by the expression

$$v^2 = 72 + 18x - 9x^2$$

i) Between which two points is the particle oscillating? **1**

ii) What is the amplitude of the motion? **1**

iii) Find the acceleration of the particle in terms of x. **1**

iv) Find the period of the oscillation. **1**

c) i) Use mathematical induction to prove $3^n > n^3$ for all integers $n \ge 4$. **3**

ii) Hence or otherwise, show that $\sqrt[3]{3} > \sqrt[n]{n}$ for all integers $n \ge 4$. **1**

End of Questions 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

a) A particle of unit mass moves in a straight line with variable acceleration $\left(\frac{16}{v} - v\right)\ ms^{-2}$ where $v\ ms^{-1}$ is the velocity at time t seconds and $v > 0$, and x is the displacement. If initially, the particle is at the origin with a velocity of $v = 2\ ms^{-1}$.

i) Find an expression for the velocity v of the particle at time t seconds. **3**

ii) Find the limiting velocity of the particle. **1**

iii) Find the displacement of the particle when $v = 3\ ms^{-1}$. **4**

b) i) Use de Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. **2**

ii) Hence show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. **1**

iii) Given $\alpha = \tan^{-1}\dfrac{1}{4}$, hence show that $4\alpha = \tan^{-1}\dfrac{240}{161}$. **1**

iv) Write $161 + 240i$ in the form $r(\cos\theta + i\sin\theta)$, express θ in terms of α. **1**

v) Hence, find in the form $a + bi$ where a and b are integers, the four fourth roots of $161 + 240i$. **2**

End of Questions 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

a) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta$, $\quad n = 0, 1, 2, \ldots$

i) Explain why $I_n < I_{n-1} < I_{n-2}$ **1**

ii) Prove that $I_n = \dfrac{n-1}{n} I_{n-2}$, $\quad n = 2, 3, 4, \ldots$ **3**

iii) Deduce that $\lim_{n\to\infty} I_n = \lim_{n\to\infty} I_{n-1}$ **1**

iv) Use part ii) to show that $n \times I_n \times I_{n-1} = \dfrac{\pi}{2}$, $\quad n = 1, 2, 3, \ldots$ **2**

v) Given that $\int_0^{\frac{\pi}{2}} \cos^{11}\theta\, d\theta = \dfrac{256}{693}$, evaluate $\int_0^{\frac{\pi}{2}} \cos^{10}\theta\, d\theta$ **1**

vi) Find an approximate value of $\int_0^{\frac{\pi}{2}} \cos^{2021}\theta\, d\theta$, to 4 significant figures. **1**

b) i) Show that for all values of a and b, $\quad \sin a \sin b \le \left[\sin\left(\dfrac{a+b}{2}\right)\right]^2$ **2**

ii) Hence show that if $\sin a > 0$, $\sin b > 0$, $\sin c > 0$, $\sin d > 0$, then **3**

$$\sin a \sin b \sin c \sin d \le \left[\sin\left(\frac{a+b+c+d}{4}\right)\right]^4$$

iii) By choosing a suitable expression for d, show that **1**

$$\sin a \sin b \sin c \le \left[\sin\left(\frac{a+b+c}{3}\right)\right]^3$$

End of Paper

Student Name:____

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word 'correct' and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

2022 Mathematics Extension 2 AT4 Trial Solutions

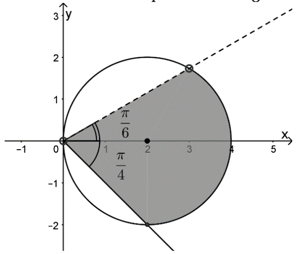
Section 1		
Q1	**C** $\lvert -3-3\sqrt{3}i \rvert = 6$ $\arg(-3-3\sqrt{3}i) = -\frac{2\pi}{3}$ $-3-3\sqrt{3}i = 6e^{-\frac{2i\pi}{3}}$	1 Mark
Q2	**A** $x = 1-3\lambda \quad (1)$ $y = 5+2\lambda \quad (2)$ From (1) $\lambda = \frac{1-x}{3}$ Sub into (2) $y = 5+2\left(\frac{1-x}{3}\right)$ $3y = 15+2-2x$ $2x+3y-17=0$	1 Mark
Q3	**D** $I = \int_1^{e^5} \frac{(\log_e x)^5}{x}dx$ Let $u = \log_e x$ $du = \frac{1}{x}dx$ $x = e^5, \quad u = 5$ $x = 1, \quad u = 0$ $I = \int_0^5 u^5 du$	1 Mark
Q4	**B** "If A, then B" statements, contrapositive is "If not B, then not A" "If x is even, then $5x-11$ is odd" Contrapositive is "If $5x-11$ is not odd, then x is not even" which is the same as "If $5x-11$ is even, then x is odd."	1 Mark
Q5	**A** $\int \sin^3 x\, dx$ $= \int \sin x\,(\sin^2 x)dx$ $= \int \sin x\,(1-\cos^2 x)dx$ $= \int \sin x\, dx - \int \sin x \cos^2 x\, dx$ $= -\cos x + \frac{1}{3}\cos^3 x + C$ $= \frac{1}{3}\cos^3 x - \cos x + C$	1 Mark
Q6	**B** $\overline{z}$ is reflected along the horizontal axis to z, then $-i\,\overline{z}$ rotate $\overline{z}$ clockwise by 90°	1 Mark

Q7	**C** If the polynomial is real, then complex roots appear in conjugate pair If one of the root is $-3+2i$, then $-3-2i$ is also a root, let the third root be γ $(-3+2i)+(-3-2i)+\gamma=-\frac{m}{5}$ $\gamma=-\frac{m}{5}+6 \quad (1)$ $(-3+2i)(-3-2i)\gamma=\frac{13}{5}$ $13\gamma=\frac{13}{5}$ $\gamma=\frac{1}{5}$ Sub into (1) $\frac{1}{5}=-\frac{m}{5}+6$ $m=29$	1 Mark
Q8	**D** Since a, b are real, and c, d are purely imaginary, then ab, cd are real, so $ab-cd$ is also real also ac, bd are purely imaginary, so $ac-bd$ is also purely imaginary $(ab-cd)^2 \geq 0$ $a^2b^2+c^2d^2 \geq 2abcd$ $(ac-bd)^2 \leq 0$ $a^2c^2+b^2d^2 \leq 2abcd$	1 Mark
Q9	**D** $\frac{dx}{dt}=x$ $\int \frac{dx}{x}=\int dt$ $\ln\|x\|=t+C$ $x=Ae^t$ At $t=5, x=-1$ $-1=Ae^5$ $A=-\frac{1}{e^5}=-e^{-5}$ $x=-e^{-5}\times e^t$ $\therefore x=-e^{t-5}$	1 Mark
Q10	**B** The functions in C and D are odd, so they both equal 0. A, the function is even but maximum value of $e^{-x^2}-2$ is -1, so the integral results in a negative value. B is the only one that is even and positive for the given boundaries.	1 Mark

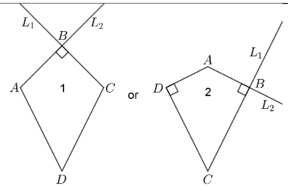
Section 2		
Q11ai	$z=\sqrt{3}-i$ $z=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$	2 Marks Correct solution 1 Mark Correct mod or arg
Q11aii	$z^7=\left(2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)\right)^7$ $z^7=2^7\left(\cos\left(-\frac{7\pi}{6}\right)+i\sin\left(-\frac{7\pi}{6}\right)\right)$ $z^7=2^7\left(\cos\left(-\frac{7\pi}{6}\right)+i\sin\left(-\frac{7\pi}{6}\right)\right)$ $z^7=128\left(-\frac{\sqrt{3}}{2}+i\times\frac{1}{2}\right)$ $z^7=-64\sqrt{3}+64i$	2 Marks Correct solution 1 Mark Correctly applies De Moivre's theorem
Q11aiii	$\frac{z^7}{(\bar{z})^7}$ $=\frac{2^7\left(\cos\left(-\frac{7\pi}{6}\right)+i\sin\left(-\frac{7\pi}{6}\right)\right)}{2^7\left(\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right)}$ $=\cos\left(-\frac{7\pi}{6}-\frac{7\pi}{6}\right)+i\sin\left(-\frac{7\pi}{6}-\frac{7\pi}{6}\right)$ $=\cos\left(-\frac{7\pi}{3}\right)+i\sin\left(-\frac{7\pi}{3}\right)$ $=\frac{1}{2}-\frac{\sqrt{3}}{2}i$	2 Marks Correct solution 1 Mark Finds $(\bar{z})^7=2^7\left(\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right)$
Q11bi	$\left\|\underset{\sim}{p}\right\|=\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2}=\sqrt{\frac{9}{9}}=1$ $\therefore \underset{\sim}{p}$ is a unit vector	1 Mark Correct solution
Q11bii	$\underset{\sim}{p}\cdot\underset{\sim}{q}=\frac{2}{3}\times\left(-\frac{1}{3}\right)+\frac{1}{3}\times\left(-\frac{2}{3}\right)+\frac{2}{3}\times\frac{2}{3}=0$ $\underset{\sim}{q}\cdot\underset{\sim}{r}=\left(-\frac{1}{3}\right)\times\frac{2}{3}+\left(-\frac{2}{3}\right)\times\left(-\frac{2}{3}\right)+\frac{2}{3}\times\left(-\frac{1}{3}\right)=0$ $\underset{\sim}{p}\cdot\underset{\sim}{r}=\frac{2}{3}\times\frac{2}{3}+\frac{1}{3}\times\left(-\frac{2}{3}\right)+\frac{2}{3}\times\left(-\frac{1}{3}\right)=0$ $\therefore \underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$ are perpendicular to each other.	2 Marks Correct solution 1 Mark Shows one pair of vectors are perpendicular
Q11biii	$\underset{\sim}{s}=\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{r}$ $\underset{\sim}{s}=\left(\frac{2}{3}\underset{\sim}{i}+\frac{1}{3}\underset{\sim}{j}+\frac{2}{3}\underset{\sim}{k}\right)+\left(-\frac{1}{3}\underset{\sim}{i}-\frac{2}{3}\underset{\sim}{j}+\frac{2}{3}\underset{\sim}{k}\right)+\left(\frac{2}{3}\underset{\sim}{i}-\frac{2}{3}\underset{\sim}{j}-\frac{1}{3}\underset{\sim}{k}\right)$ $\underset{\sim}{s}=\left(\frac{2}{3}-\frac{1}{3}+\frac{2}{3}\right)\underset{\sim}{i}+\left(\frac{1}{3}-\frac{2}{3}-\frac{2}{3}\right)\underset{\sim}{j}+\left(\frac{2}{3}+\frac{2}{3}-\frac{1}{3}\right)\underset{\sim}{k}$ $\underset{\sim}{s}=\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$	1 Mark Correct solution
Q11biv	$\underset{\sim}{s}\cdot\underset{\sim}{t}=\left(\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{r}\right)\cdot\left(\frac{2}{\sqrt{3}}\underset{\sim}{p}-\sqrt{3}\underset{\sim}{q}+\frac{1}{\sqrt{3}}\underset{\sim}{r}\right)$ $\underset{\sim}{s}\cdot\underset{\sim}{t}=1\times\frac{2}{\sqrt{3}}+1\times(-\sqrt{3})+1\times\frac{1}{\sqrt{3}}$ $\underset{\sim}{s}\cdot\underset{\sim}{t}=0$	1 Mark Correct solution

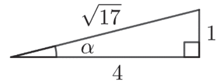
Q11c	$\frac{2-x}{(1+5x^2)(1+10x)}=\frac{A+Bx}{1+5x^2}+\frac{C}{1+10x}$ $(A+Bx)(1+10x)+C(1+5x^2)=2-x$ Let $x=-\frac{1}{10}$ $(A+Bx)\times 0+C\left(1+5\times\left(-\frac{1}{10}\right)^2\right)=2-\left(-\frac{1}{10}\right)$ $C=2$ Let $x=0$ $(A+B\times 0)(1+10\times 0)+2(1+5\times 0^2)=2-0$ $A+2=2$ $A=0$ Let $x=1$ $(0+B\times 1)(1+10\times 1)+2(1+5\times 1^2)=2-1$ $11B+12=1$ $B=-1$ $\int_0^1\frac{2-x}{(1+5x^2)(1+10x)}dx$ $=\int_0^1\frac{-x}{1+5x^2}+\frac{2}{1+10x}dx$ $=\left[-\frac{1}{10}\log_e\lvert 1+5x^2\rvert+\frac{1}{5}\log_e\lvert 1+10x\rvert\right]_0^1$ $=-\frac{1}{10}\log_e\lvert 1+5\times 1^2\rvert+\frac{1}{5}\log_e\lvert 1+10\times 1\rvert$ $-\left(-\frac{1}{10}\log_e\lvert 1+0\rvert+\frac{1}{5}\log_e\lvert 1+0\rvert\right)$ $=-\frac{1}{10}\log_e 6+\frac{1}{5}\log_e 11$ $=-\frac{1}{10}\log_e 6+\frac{1}{10}\log_e 11^2$ $=\frac{1}{10}\log_e\frac{121}{6}$	4 Marks Correct solution 3 Marks Correct integration 2 Marks Correct partial fraction 1 Mark Finds the correct value of A, B or C
Q12ai	$I=\int\frac{1}{x}(1+\log_e x)^6dx$ Let $u=1+\log_e x$ $du=\frac{1}{x}dx$ $I=\int u^6du$ $I=\frac{u^7}{7}+C$ $I=\frac{(1+\log_e x)^7}{7}+C$	2 Marks Correct solution 1 Mark Correct substitution
Q12aii	$\int\frac{1}{\sqrt{2+x}+\sqrt{x}}dx$ $=\int\frac{1}{\sqrt{2+x}+\sqrt{x}}\times\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}dx$ $=\int\frac{\sqrt{2+x}-\sqrt{x}}{2+x-x}dx$ $=\frac{1}{2}\int\left((2+x)^{\frac{1}{2}}-x^{\frac{1}{2}}\right)dx$	2 Marks Correct solution 1 Mark Correctly rationalises the denominator

	$=\frac{1}{2}\left[\frac{(2+x)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]+C$ $=\frac{1}{3}\left[(2+x)^{\frac{3}{2}}-x^{\frac{3}{2}}\right]+C$	
Q12aiii	$I=\int\sin^3 x\cos^2 x\,dx$ $=\int(1-\cos^2 x)\cos^2 x\sin x\,dx$ $=\int(\cos^2 x-\cos^4 x)\sin x\,dx$ Let $u=\cos x$, $du=-\sin x\,dx$ $I=-\int(u^2-u^4)du$ $I=-\frac{u^3}{3}+\frac{u^5}{5}+C$ $I=-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$	3 Marks Correct solution 2 Marks Correct integration in terms of u 1 Mark Obtains $\int(1-\cos^2 x)\cos^2 x\sin x\,dx$
Q12aiv	$\int\frac{2x+7}{x^2+10x+29}dx$ $=\int\frac{2x+10}{x^2+10x+29}-\frac{3}{x^2+10x+25+4}dx$ $=\int\frac{2x+10}{x^2+10x+29}-\frac{3}{(x+5)^2+2^2}dx$ $=\log_e\lvert x^2+10x+29\rvert-\frac{3}{2}\tan^{-1}\frac{(x+5)}{2}+C$	3 Marks Correct solution 2 Marks Completes the square and makes significant progress 1 Mark Correct split of the numerator
Q12bi	Since $P(x)$ is real, then complex roots appears in conjugate pairs. If $x=\frac{1-i\sqrt{3}}{2}$ is a root, then $x=\frac{1+i\sqrt{3}}{2}$ is also a root. The roots are $7,\frac{1-i\sqrt{3}}{2},\frac{1+i\sqrt{3}}{2},\alpha$ $P(x)=(x-7)(x-\alpha)\left(x-\frac{1-i\sqrt{3}}{2}\right)\left(x-\frac{1+i\sqrt{3}}{2}\right)$ $\frac{1-i\sqrt{3}}{2}+\frac{1+i\sqrt{3}}{2}=1=-\frac{b}{a}$ $\left(\frac{1-i\sqrt{3}}{2}\right)\left(\frac{1+i\sqrt{3}}{2}\right)=1=\frac{c}{a}$ $P(x)=(x-7)(x-\alpha)(x^2-x+1)$ $\therefore (x^2-x+1)$ is a factor of $P(x)$.	2 Marks Correct solution 1 Mark Identifies $x=\frac{1+i\sqrt{3}}{2}$
Q12bii	$P(x)=(x^2-(7+\alpha)x+7\alpha)(x^2-x+1)$ $P(x)=x^4+ax^3-40x^2+41x+b$ Compare the coefficient of the x term $-(7+\alpha)\times 1+7\alpha\times(-1)=41$ $-7-\alpha-7\alpha=41$ $-8\alpha=48$ $\alpha=-6$	3 Marks Correct solution 2 Marks Finds $x=-6$ as a root and either a or b 1 Mark Finds $x=-6$ as a root

	Sum of roots $-6+7+\frac{1-i\sqrt{3}}{2}+\frac{1+i\sqrt{3}}{2}=2$ $a=-2$ Product of roots $-6\times 7\times\left(\frac{1-i\sqrt{3}}{2}\right)\left(\frac{1+i\sqrt{3}}{2}\right)=-42$ $b=-42$	
Q13a	$\lvert z-2\rvert\le 2\ \cap\ -\frac{\pi}{4}\le\arg z<\frac{\pi}{6}$ 	3 Marks Correct solution 2 Marks Correct region with most key features shown 1 Mark Correct circle or argument
Q13b	$\left(6\underset{\sim}{i}+p\underset{\sim}{j}-13\underset{\sim}{k}\right)\cdot\left((1-4p)\underset{\sim}{i}+(p+3)\underset{\sim}{j}+6\underset{\sim}{k}\right)=0$ $6(1-4p)+p(p+3)-13\times 6=0$ $6-24p+p^2+3p-78=0$ $p^2-21p-72=0$ $(p-24)(p+3)=0$ $p=-3,24$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Establishes $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$
Q13c	$F=m\ddot{x}$ $m\ddot{x}=\frac{m}{x^3}(4-7x)$ $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=4x^{-3}-7x^{-2}$ $\frac{1}{2}v^2=\int(4x^{-3}-7x^{-2})dx$ $v^2=2\left[\frac{4x^{-2}}{-2}-\frac{7x^{-1}}{-1}\right]+C$ $v^2=-\frac{4}{x^2}+\frac{14}{x}+C$ $v^2=\frac{14x-4+Cx^2}{x^2}$ $v=\pm\frac{1}{x}\sqrt{Cx^2+14x-4}$ $v=-1$ when $x=1$, $\therefore v=-\frac{1}{x}\sqrt{Cx^2+14x-4}$ $-1=-\frac{1}{1}\sqrt{C\times 1^2+14\times 1-4}$ $1=\sqrt{C+10}$ $C=-9$ $\therefore v=-\frac{1}{x}\sqrt{-9x^2+14x-4}$	3 Marks Correct solution 2 Marks Finds $v=\pm\frac{1}{x}\sqrt{14x-4+Cx^2}$ 1 Mark Finds $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=4x^{-3}-7x^{-2}$

Q13d	$I=\int e^x\cos 10x\,dx$ $u=\cos 10x \qquad v'=e^x$ $u'=-10\sin 10x \qquad v=e^x$ $I=e^x\cos 10x+10\int e^x\sin 10x\,dx$ $u=\sin 10x \qquad v'=e^x$ $u'=10\cos 10x \qquad v=e^x$ $I=e^x\cos 10x+10\left(e^x\sin 10x-\int 10e^x\cos 10x\,dx\right)$ $I=e^x\cos 10x+10e^x\sin 10x-100I+C$ $101I=e^x\cos 10x+10e^x\sin 10x+C$ $I=\frac{e^x\cos 10x+10e^x\sin 10x}{101}+C$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Correctly applies integration by parts
Q13ei	$f(x)=\sqrt{1+x}$ $f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}=\frac{1}{2\sqrt{1+x}}$ $f'(x)=\frac{1}{2\sqrt{1+x}}$ Since $x>8$ $\sqrt{1+x}>\sqrt{1+8}$ $\sqrt{1+x}>3$ $2\sqrt{1+x}>6$ $\frac{1}{2\sqrt{1+x}}<\frac{1}{6}$ $\therefore f'(x)<\frac{1}{6},\quad x>8$	1 Mark Correct solution
Q13eii	Let $g(x)=3+\frac{x-8}{6}$ $g'(x)=\frac{1}{6}$ $f'(x)<\frac{1}{6},\quad x>8$ $\therefore f'(x)<g'(x),\quad x>8$ Also $g(8)=3=f(8)$ $\therefore f(x)\le g(x),\quad x\ge 8$ $\therefore \sqrt{1+x}\le 3+\frac{x-8}{6},\quad x\ge 8$	2 Marks Correct solution 1 Mark Shows $f'(x)<g'(x)$
Q14ai	$\underset{\sim}{r_1}=\begin{bmatrix}-1\\5\\-1\end{bmatrix}+\lambda\begin{bmatrix}1\\-2\\1\end{bmatrix},\underset{\sim}{r_2}=\begin{bmatrix}-3\\1\\5\end{bmatrix}+\mu\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\lambda=0,\ \underset{\sim}{a}=\begin{bmatrix}-1\\5\\-1\end{bmatrix},\quad \mu=0,\ \underset{\sim}{c}=\begin{bmatrix}-3\\1\\5\end{bmatrix}$ $\overrightarrow{AC}=\underset{\sim}{c}-\underset{\sim}{a}=\begin{bmatrix}-2\\-4\\6\end{bmatrix}$ $\lvert\overrightarrow{AC}\rvert=\sqrt{(-2)^2+(-4)^2+6^2}$ $\lvert\overrightarrow{AC}\rvert=\sqrt{56}$ $\lvert\overrightarrow{AC}\rvert=2\sqrt{14}$	2 Marks Correct solution 1 Mark Finds $\overrightarrow{AC}$

Q14aii	At the point of intersection $\underset{\sim}{r_1} = \underset{\sim}{r_2}$ Equate $\underset{\sim}{j}$ $5 - 2\lambda = 1 + 0$ $2\lambda = 4$ $\lambda = 2$ Equate $\underset{\sim}{i}$ $-1 + \lambda = -3 + \mu$ $\lambda + 2 = \mu$ $\mu = 4$ Check $\underset{\sim}{k}$ $-1 + 2 \times 1 = 1 = 5 + 4 \times -1$ $\therefore$ The lines intersect $\therefore B = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	2 Marks Correct solution 1 Mark1 Finds the correct λ or μ
Q14aiii	$\cos\theta = \dfrac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\sqrt{1^2 + (-2)^2 + 1^2} \times \sqrt{1^2 + 0^2 + (-1)^2}}$ $\cos\theta = \dfrac{1 \times 1 + (-2) \times 0 + 1 \times (-1)}{\sqrt{6} \times \sqrt{2}}$ $\cos\theta = 0$ $\theta = 90°$	1 Mark Correct solution
Q14aiv	 $\overrightarrow{AB} = \begin{bmatrix} 1 - (-1) \\ 1 - 5 \\ 1 - (-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$ $\lvert\overrightarrow{AB}\rvert = \sqrt{2^2 + (-4)^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$ $\overrightarrow{BC} = \begin{bmatrix} -3 - 1 \\ 1 - 1 \\ 5 - 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$ $\lvert\overrightarrow{BC}\rvert = \sqrt{(-4)^2 + 0^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$ $\lvert\overrightarrow{AB}\rvert \neq \lvert\overrightarrow{BC}\rvert$ $\therefore$ It is kite 2 Area of kite $= 2 \times$ triangles $A = 2 \times \frac{1}{2} \times \lvert\overrightarrow{AB}\rvert \times \lvert\overrightarrow{BC}\rvert$ $A = 2\sqrt{6} \times 4\sqrt{2}$ $A = 16\sqrt{3}$ units^2	2 Marks Correct solution 1 Mark Makes significant progress
Q14bi	$v^2 = 72 + 18x - 9x^2$ $72 + 18x - 9x^2 = 0$ $-9(x^2 - 2x - 8) = 0$ $-9(x - 4)(x + 2) = 0$ $x = -2, x = 4$ $\therefore$ Particle oscillates between $x = -2$ and $x = 4$.	1 Mark Correct solution
Q14bii	$\frac{1}{2} \times (4 - (-2)) = 3$ Amplitude is 3	1 Mark Correct solution
Q14biii	$v^2 = 72 + 18x - 9x^2$ $\frac{1}{2}v^2 = 36 + 9x - \frac{9}{2}x^2$ $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $\ddot{x} = 9 - 9x$	1 Mark Correct solution
Q14biv	$\ddot{x} = 9 - 9x$ $\ddot{x} = -3^2(x - 1)$ $\therefore n = 3$ $T = \frac{2\pi}{n}$ $T = \frac{2\pi}{3}$	1 Mark Correct solution
Q14ci	RTP: $3^n > n^3$, for all integers $n \geq 4$ 1. Prove statement is true for $n = 4$ $LHS = 3^4 = 81$ $RHS = 4^3 = 64$ $LHS > RHS$ $\therefore$statement is true for $n = 4$ 2. Assume statement is true for $n = k$, where $k \geq 4$ i.e. $3^k > k^3 \to 3^k - k^3 > 0$ 3. Prove statement is true for $n = k + 1$ i.e. $3^{k+1} > (k + 1)^3 \to 3^{k+1} - (k + 1)^3 > 0$ Consider $3^{k+1} - (k + 1)^3$ $3^{k+1} - (k + 1)^3$ $= 3 \times 3^k - (k^3 + 3k^2 + 3k + 1)$ $= 3 \times 3^k - k^3 - 3k^2 - 3k - 1$ $= 3(3^k - k^3) + 2k^3 - 3k^2 - 3k - 1$ $= 3(3^k - k^3) + (k^3 - 3k^2 + 3k - 1) + k^3 - 6k$ $= 3(3^k - k^3) + (k - 1)^3 + k(k^2 - 6)$ $3^k - k^3 > 0$ from assumption, so $3(3^k - k^3) > 0$ $k - 1 > 0$ as $k \geq 4$, so $(k - 1)^3 > 0$ $k^2 - 6 > 0$ as $k \geq 4$, so $k(k^2 - 6) > 0$ Then $3(3^k - k^3) + (k - 1)^3 + k(k^2 - 6) > 0$ $3^{k+1} - (k + 1)^3 > 0$ $\therefore 3^{k+1} > (k + 1)^3$ By mathematical induction, $3^n > n^3$, for all integers $n \geq 4$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Proves the initial statement
Q14cii	$3^n > n^3$ $(3^n)^{\frac{1}{3n}} > (n^3)^{\frac{1}{3n}}$ $3^{\frac{1}{3}} > n^{\frac{1}{n}}$ $\therefore \sqrt[3]{3} > \sqrt[n]{n}$	1 Mark Correct solution

Q15ai	$\ddot{x} = \left(\frac{16}{v} - v\right)$ $\frac{dv}{dt} = \frac{16 - v^2}{v}$ $\frac{dt}{dv} = \frac{v}{16 - v^2}$ $t = -\frac{1}{2}\log_e\lvert 16 - v^2\rvert + C$ $2C - 2t = \log_e\lvert 16 - v^2\rvert$ $Ae^{-2t} = 16 - v^2$ $v^2 = 16 - Ae^{-2t}$ $v = \pm\sqrt{16 - Ae^{-2t}}$ Since $v > 0$, then $v = \sqrt{16 - Ae^{-2t}}$ $t = 0, v = 2$ $2 = \sqrt{16 - A}$ $A = 12$ $\therefore v = \sqrt{16 - 12e^{-2t}}$	3 Marks Correct solution 2 Marks Makes significant progress or finds $C = \frac{1}{2}ln12$ 1 Mark Correct primitive function of t in terms of v
Q15aii	$t \to \infty, 12e^{-2t} \to 0$ $v \to \sqrt{16 - 0}$ $v \to 4\ m/s$	1 Mark Correct solution
Q15aiii	$v\frac{dv}{dx} = \frac{16}{v} - v$ $\frac{dv}{dx} = \frac{16}{v^2} - 1$ $\frac{dx}{dv} = \frac{v^2}{16 - v^2}$ $\frac{dx}{dv} = \frac{-(16 - v^2) + 16}{16 - v^2}$ $\frac{dx}{dv} = \frac{16}{16 - v^2} - 1$ $\frac{dx}{dv} = \left(\frac{2}{4 - v} + \frac{2}{4 + v}\right) - 1$ $x = 2\log_e\lvert 4 + v\rvert - 2\log_e\lvert 4 - v\rvert - v + C$ $x = 2\log_e\left\lvert\frac{4 + v}{4 - v}\right\rvert - v + C$ $x = 0, v = 2$ $0 = 2\log_e\left\lvert\frac{4 + 2}{4 - 2}\right\rvert - 2 + C$ $C = 2 - 2\log_e 3$ $\therefore x = 2\log_e\left\lvert\frac{4 + v}{4 - v}\right\rvert - v + 2 - 2\log_e 3$ $v = 3$ $x = 2\log_e\left\lvert\frac{4 + 3}{4 - 3}\right\rvert - 3 + 2 - 2\log_e 3$ $x = 2\log_e 7 - 1 - 2\log_e 3$ $x = 2\log_e\left(\frac{7}{3}\right) - 1$	4 Marks Correct solution 3 Marks Finds the correct value of C 2 Marks Finds $\frac{dx}{dv} = \left(\frac{2}{4 - v} + \frac{2}{4 + v}\right) - 1$ 1 Mark Finds $\frac{dv}{dx} = \frac{16}{v^2} - 1$
Q15bi	$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ (de Moivre's theorem) $= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$ Equate real: $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ Equate imaginary: $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$	2 Marks Correct solution 1 Mark Equate either real or imaginary
Q15bii	$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$ $\tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \div \frac{\cos^4\theta}{\cos^4\theta}$ $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$	1 Mark Correct solution
Q15biii	$\alpha = \tan^{-1}\frac{1}{4}$ $\tan\alpha = \frac{1}{4}$ $\tan 4\alpha = \frac{4 \times \left(\frac{1}{4}\right) - 4 \times \left(\frac{1}{4}\right)^3}{1 - 6 \times \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^4}$ $\tan 4\alpha = \frac{240}{161}$ $4\alpha = \tan^{-1}\frac{240}{161}$	1 Mark Correct solution
Q15biv	$\lvert 161 + 240i\rvert = 289$ $\arg(161 + 240i) = \tan^{-1}\frac{240}{161} = 4\alpha$ $161 + 240i = 289(\cos 4\alpha + i\sin 4\alpha)$	1 Mark Correct solution
Q15bv	Let $z^4 = 161 + 240i$ $z^4 = 289\,\text{cis}(4\alpha + 2k\pi)$ where $n \in \mathbb{Z}$ $z = 289^{\frac{1}{4}}\,\text{cis}\left(\alpha + \frac{k\pi}{2}\right)$ $z = \sqrt{17}\,\text{cis}\left(\alpha + \frac{k\pi}{2}\right)$ $k = 0,\ z_1 = \sqrt{17}\,\text{cis}\,\alpha$ $k = 1,\ z_2 = \sqrt{17}\,\text{cis}\left(\alpha + \frac{\pi}{2}\right)$ $k = 2,\ z_3 = \sqrt{17}\,\text{cis}(\alpha + \pi)$ $k = 3,\ z_4 = \sqrt{17}\,\text{cis}\left(\alpha + \frac{3\pi}{2}\right) = \sqrt{17}\,\text{cis}\left(\alpha - \frac{\pi}{2}\right)$ $\tan\alpha = \frac{1}{4}$  $\sin\alpha = \frac{1}{\sqrt{17}},\ \cos\alpha = \frac{4}{\sqrt{17}}$ $z_1 = \sqrt{17}(\cos\alpha + i\sin\alpha) = \sqrt{17}\left(\frac{4}{\sqrt{17}} + i\frac{1}{\sqrt{17}}\right)$ $z_1 = 4 + i$ $z_2 = \sqrt{17}\left(\cos\left(\alpha + \frac{\pi}{2}\right) + i\sin\left(\alpha + \frac{\pi}{2}\right)\right) = \sqrt{17}\left(-\frac{1}{\sqrt{17}} + i\frac{4}{\sqrt{17}}\right)$ $z_2 = -1 + 4i$ $z_3 = \sqrt{17}(\cos(\alpha + \pi) + i\sin(\alpha + \pi)) = \sqrt{17}\left(-\frac{4}{\sqrt{17}} - i\frac{1}{\sqrt{17}}\right)$ $z_3 = -4 - i$ $z_4 = \sqrt{17}\left(\cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right)\right) = \sqrt{17}\left(\frac{1}{\sqrt{17}} - i\frac{4}{\sqrt{17}}\right)$ $z_4 = 1 - 4i$ $\therefore$ the roots $4 + i,\ -1 + 4i,\ -4 - i,\ 1 - 4i$	2 Marks Correct solution 1 Mark Makes significant progress and finds one of the solution in $a + ib$ form

Q16ai	$0 < x < \frac{\pi}{2}, \quad 0 < \cos x < 1$ $\cos^n x$, as $n \to \infty$, $\cos^n x \to 0^+$ $\therefore \cos^n x < \cos^{n-1} x < \cos^{n-2} x$ $\therefore I_n < I_{n-1} < I_{n-2}$	1 Mark Correct solution
Q16aii	$I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos\theta \cos^{n-1}\theta \, d\theta$ $u = \cos^{n-1}\theta \qquad v' = \cos\theta$ $u' = -(n-1)\sin\theta\cos^{n-2}\theta \qquad v = \sin\theta$ $I_n = [\sin\theta\cos^{n-1}\theta]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^{n-2}\theta \, d\theta$ $I_n = \left(\sin\frac{\pi}{2} \times \cos^{n-1}\frac{\pi}{2} - \sin 0 \times \cos^{n-1} 0\right)$ $\qquad + (n-1)\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^{n-2}\theta \, d\theta$ $I_n = (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\cos^{n-2}\theta \, d\theta$ $I_n = (n-1)\int_0^{\frac{\pi}{2}} (\cos^{n-2}\theta - \cos^n\theta) d\theta$ $I_n = (n-1)(I_{n-2} - I_n)$ $I_n = (n-1)I_{n-2} - (n-1)I_n$ $I_n + (n-1)I_n = (n-1)I_{n-2}$ $(1+n-1)I_n = (n-1)I_{n-2}$ $I_n = \frac{n-1}{n} I_{n-2}$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Integrate by parts
Q16aiii	$I_n = \left(1 - \frac{1}{n}\right) I_{n-2}$ $n \to \infty, \quad \frac{1}{n} \to 0, \quad 1 - \frac{1}{n} \to 1$ $\therefore \lim_{n\to\infty} I_n = \lim_{n\to\infty} I_{n-2}$ From part i, $\quad I_n < I_{n-1} < I_{n-2}$ $\therefore \lim_{n\to\infty} I_n = \lim_{n\to\infty} I_{n-1}$	1 Mark Correct solution
Q16aiv	From part ii, $I_n = \frac{n-1}{n} I_{n-2}$ $n \times I_n \times I_{n-1}$ $= n \times \frac{n-1}{n} I_{n-2} \times \frac{n-2}{n-1} I_{n-3}$ $= (n-2) \times I_{n-2} \times I_{n-3}$ $= \cdots$ $= 1 \times I_1 \times I_0$ $= \int_0^{\frac{\pi}{2}} (\cos\theta)^1 d\theta \times \int_0^{\frac{\pi}{2}} (\cos\theta)^0 d\theta$ $= \int_0^{\frac{\pi}{2}} (\cos\theta) d\theta \times \int_0^{\frac{\pi}{2}} 1 d\theta$ $= [\sin\theta]_0^{\frac{\pi}{2}} \times [\theta]_0^{\frac{\pi}{2}}$ $= \left(\sin\frac{\pi}{2} - \sin 0\right) \times \left(\frac{\pi}{2} - 0\right)$ $= \frac{\pi}{2}$	2 Marks Correct solution 1 Mark Deduces $n \times I_n \times I_{n-1}$ $= (n-2) \times I_{n-2} \times I_{n-3}$

Q16av	Given that $I_{11} = \int_0^{\frac{\pi}{2}} \cos^{11}\theta \, d\theta = \frac{256}{693}$ From part iv $11 \times I_{11} \times I_{10} = \frac{\pi}{2}$ $I_{10} = \frac{\pi}{2} \div 11 \div \frac{256}{693}$ $I_{10} = \frac{63\pi}{512}$	1 Mark Correct solution
Q16avi	From part iii and iv, as $n \to \infty$, $n \times I_n \times I_{n-1} \approx n \times (I_n)^2$ $2021 \times (I_{2021})^2 \approx \frac{\pi}{2}$ $I_{2021} \approx \sqrt{\frac{\pi}{4042}}$ $I_{2021} \approx 0.0278789 \ldots$ $I_{2021} = 0.02788$ (4 sig. fig)	1 Mark Correct solution
Q16bi	$\left[\sin\left(\frac{a+b}{2}\right)\right]^2 - \sin a \sin b$ $= \left[\sin\frac{a}{2}\cos\frac{b}{2} + \sin\frac{b}{2}\cos\frac{a}{2}\right]^2 - \sin\left(2 \times \frac{a}{2}\right)\sin\left(2 \times \frac{b}{2}\right)$ $= \left(\sin^2\frac{a}{2}\cos^2\frac{b}{2} + 2\sin\frac{a}{2}\cos\frac{b}{2}\sin\frac{b}{2}\cos\frac{a}{2} + \sin^2\frac{b}{2}\cos^2\frac{a}{2}\right)$ $\qquad - 2\sin\frac{a}{2}\cos\frac{a}{2} \times 2\sin\frac{b}{2}\cos\frac{b}{2}$ $= \sin^2\frac{a}{2}\cos^2\frac{b}{2} + \sin^2\frac{b}{2}\cos^2\frac{a}{2} + 2\sin\frac{a}{2}\cos\frac{b}{2}\sin\frac{b}{2}\cos\frac{a}{2}$ $\qquad - 4\sin\frac{a}{2}\cos\frac{b}{2}\sin\frac{b}{2}\cos\frac{a}{2}$ $= \sin^2\frac{a}{2}\cos^2\frac{b}{2} + \sin^2\frac{b}{2}\cos^2\frac{a}{2} - 2\sin\frac{a}{2}\cos\frac{b}{2}\sin\frac{b}{2}\cos\frac{a}{2}$ $= \left[\sin\frac{a}{2}\cos\frac{b}{2} - \sin\frac{b}{2}\cos\frac{a}{2}\right]^2$ $= \left[\sin\left(\frac{a-b}{2}\right)\right]^2$ $\left[\sin\left(\frac{a-b}{2}\right)\right]^2 \geq 0$ $\left[\sin\left(\frac{a+b}{2}\right)\right]^2 - \sin a \sin b \geq 0$ $\therefore \sin a \sin b \leq \left[\sin\left(\frac{a+b}{2}\right)\right]^2$	2 Marks Correct solution 1 Mark Makes significant progress
Q16bii	$\sin a \sin b \leq \left[\sin\left(\frac{a+b}{2}\right)\right]^2$ $\sin c \sin d \leq \left[\sin\left(\frac{c+d}{2}\right)\right]^2$	3 Marks Correct solution 2 Marks Makes significant progress

	$\sin a \sin b \sin c \sin d \leq \left[\sin\left(\frac{a+b}{2}\right)\right]^2 \left[\sin\left(\frac{c+d}{2}\right)\right]^2$ $\sin a \sin b \sin c \sin d \leq \left[\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{c+d}{2}\right)\right]^2$ $\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{c+d}{2}\right) \leq \left[\sin\left(\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2}\right)\right]^2$ $\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{c+d}{2}\right) \leq \left[\sin\left(\frac{a+b+c+d}{4}\right)\right]^2$ $\left[\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{c+d}{2}\right)\right]^2 \leq \left[\left[\sin\left(\frac{a+b+c+d}{4}\right)\right]^2\right]^2$ $\left[\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{c+d}{2}\right)\right]^2 \leq \left[\sin\left(\frac{a+b+c+d}{4}\right)\right]^4$ $\therefore \sin a \sin b \sin c \sin d \leq \left[\sin\left(\frac{a+b+c+d}{4}\right)\right]^4$	1 Mark Obtains $\sin a \sin b \sin c \sin d$ $\leq \left[\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{c+d}{2}\right)\right]^2$
Q16biii	$Let\ d = \frac{1}{3}(a+b+c)$ $\sin a \sin b \sin c \sin\left(\frac{a+b+c}{3}\right)$ $\leq \left[\sin\left(\frac{a+b+c+\frac{a+b+c}{3}}{4}\right)\right]^4$ $\sin a \sin b \sin c \sin\left(\frac{a+b+c}{3}\right)$ $\leq \left[\sin\left(\frac{\frac{3a+3b+3c}{3}+\frac{a+b+c}{3}}{4}\right)\right]^4$ $\sin a \sin b \sin c \sin\left(\frac{a+b+c}{3}\right) \leq \left[\sin\left(\frac{a+b+c}{3}\right)\right]^4$ $\therefore \sin a \sin b \sin c \leq \left[\sin\left(\frac{a+b+c}{3}\right)\right]^3$	1 Mark Correct solution